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Unsharpness, Naimark Theorem and Informational Equivalence of Quantum Observables

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Abstract For each commutative POV measure F there exists (Beneduci, J. Math. Phys. 47:062104-1, 2006; Int. J. Geom. Methods Mod. Phys. 3:1559, 2006) a PV measure E such that F can be interpreted as a random diffusion of E. In its turn, the self-adjoint operator $A = \int \lambda \, dE_\lambda$ corresponding to E, can be interpreted (Beneduci, J. Math. Phys. 48:022102-1, 2007; Nuovo Cimento B 123:43–62, 2008) as the projection of a Naimark operator corresponding to the Naimark dilation E^+ of F. Moreover E can be algorithmically reconstructed by F. All that suggests that, in some sense, the observables represented by E and F should have the same informational content. We introduce an equivalence relation on the set of observables which we compare with other well known equivalence relations and prove that it is the only one for which E is always equivalent to F.

Keywords POV measures · Naimark theorem · Informational content

1 Introduction

The present paper focuses on the relationships between some new results [3–6] in the framework of unsharp quantum theory [1, 8, 11, 15, 20, 23] and the concept of informational equivalence between observables. The results quoted above establish some relationships between a given commutative POV measure F, its Neumark extension E^+ and its sharp reconstruction E. In particular in [4] it is shown that, for each real commutative POV measure F, there exists a real PV measure E (the sharp reconstruction of F), and a Markov kernel $\mu_{(\cdot)}(\cdot) : \mathcal{B}(\mathbb{R}) \times \mathbb{R} \to [0, 1]$, such that,

$$F(\Delta) = \int \mu_{(\Delta)}(\lambda) \, dE_\lambda = \mu_\Delta(A), \qquad \Delta \in \mathcal{B}(\mathbb{R}) \tag{1}$$

where, A is the self-adjoint operator corresponding to E.

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In its turn [4–6], in the discrete case, A can be interpreted as the projection of a Neumark operator corresponding to the Neumark extension E^+ of F. In particular (Theorem 3), there exist two real, bounded, one-to-one, measurable functions f and G_f such that $P^+ f(A^+)_{|\mathcal{H}} = G_f(A)$ where, A^+ is the self-adjoint operator corresponding to E^+. This gives a notion of equivalence between sharp reconstructions and projections of Neumark operators (see Definition 3).

Moreover, it is worth remarking that the sharp reconstruction E of F can be constructed by an algorithmic procedure [10] starting from F. All that suggests [2, 3, 10] that the information contained in the observable represented by F is, in some sense, equivalent to the information contained in the observable represented by its sharp reconstruction E (in other words, during the randomization process there is no loss of information). We introduce (Definition 13) a formal expression for the informational content of an observable which, in the particular case of commutative POV measures, capture the meaning described above. This introduces a partial ordering on the set of observables which we compare with other well known partial order relations, and prove that it is the only one for which the sharp reconstruction E of F is always equivalent to F.

2 Naimark Operators and Sharp Reconstructions

In the present section, we recall some basic concepts about POV measures. In what follows, $\mathcal{B}(\mathbb{R})$ and $\mathcal{F}(\mathcal{H})$, denote respectively the Borel σ-algebra of the reals and the space of bounded positive linear operators on the Hilbert space $\mathcal{H}$. The identity and the null operators are denoted by $\mathbf{1}$ and $\mathbf{0}$ respectively.

Definition 1 A Positive Operator Valued measure (for short, POV measure) is a map $F : \mathcal{B}(\mathbb{R}) \to \mathcal{F}(\mathcal{H})$ such that, for every countable family of disjoint sets $\{\Delta_n\}$ in $\mathcal{B}(\mathbb{R})$,

$$F\left(\bigcup_{n=1}^{\infty} \Delta_n\right) = \sum_{n=1}^{\infty} F(\Delta_n), \tag{2}$$

where, the series converges in the weak operator topology.

A POV measure is said normalized if $F(\mathbb{R}) = \mathbf{1}$, commutative if $[F(\Delta_1), F(\Delta_2)] = \mathbf{0}$, for any $\Delta_1, \Delta_2 \in \mathcal{B}(\mathbb{R})$, orthogonal if $F(\Delta_1)F(\Delta_2) = \mathbf{0}$ whenever $\Delta_1 \cap \Delta_2 = \emptyset$.

Definition 2 A Projection Valued measure (for short, PV measure) is an orthogonal, normalized POV measure.

In quantum mechanics, non-orthogonal normalized POV measures are also called *unsharp* observables and PV measures *sharp* observables. In the following, we shall always refer to normalized POV measures defined on $\mathcal{B}(\mathbb{R})$ and we shall not distinguish between an observable and the corresponding POV measure.

The range of a POV measure F is the set $\mathcal{R}(F) := \{F(\Delta),\ \Delta \in \mathcal{B}(\mathbb{R})\}$. We shall use the term "measurable" for the Borel measurable functions. We shall say that a measurable function $f : N \subset \mathbb{R} \to f(N) \subset \mathbb{R}$ is almost everywhere (a.e.) one-to-one with respect to a POV measure F if it is one-to-one on a subset $N' \subset N$ such that $N - N'$ is a null set with respect to F. We shall say that a function $f : \mathbb{R} \to \mathbb{R}$ is bounded with respect to a POV measure F, if it is equal to a bounded function g a.e. with respect to F, that is, if $f = g$ a.e. with respect to the measure $\langle F(\cdot)x, x\rangle$, $\forall x \in \mathcal{H}$.

For any real, bounded measurable function f and for any POV measure F, there is a unique [7] bounded self-adjoint operator B such that, $\langle Bx, x\rangle = \int f(\lambda)\, d\langle F_\lambda x, x\rangle$, for each $x \in \mathcal{H}$. In this case, we write $B = \int f(\lambda)\, dF_\lambda$ or $B = \int f(\lambda) F(d\lambda)$ equivalently.

Definition 3 Two bounded self-adjoint operators A and B are said to be equivalent if there exists a bounded, one-to-one, measurable function f such that $A = f(B)$. In this case we write $A \leftrightarrow B$.

In the following, the symbol $\mu_{(\cdot)}(\lambda)$ denotes a family (with respect to the parameter λ) of probability measures on $\mathcal{B}(\mathbb{R})$ while the symbol $\omega_{(\cdot)}(\lambda)$ denotes a family of set functions [19]. Moreover, we assume that, for every $\Delta \in \mathcal{B}(\mathbb{R})$, the functions $\mu_{(\Delta)}(\lambda)$ and $\omega_{(\Delta)}(\lambda)$ are measurable. Therefore, $\mu_{(\cdot)}(\lambda)$ is a Markov kernel.

Next, we summarize some basic results on the characterization of commutative POV measures, which will be used in the paper.

Definition 4 We say that $(F, B, \omega^B_{(\cdot)}(\lambda))$ is a von Neumann triplet if $\omega^B_{(\Delta)}(B) = F(\Delta)$, for every $\Delta \in \mathcal{B}(\mathbb{R})$.

The use of this terminology is inspired by von Neumann's theorem [22]. The following theorem characterizes the commutative POV measures.

Theorem 1 ([3, 6]) *For each commutative POV measure F, there exists a couple $(A, \mu^A_{(\cdot)}(\lambda))$ such that*: (i) $(F, A, \mu^A_{(\cdot)}(\lambda))$ *is a von Neumann triplet*; (ii) *for every von Neumann triplet $(F, B, \omega^B_{(\cdot)}(\lambda))$, there exists a real function g such that $A = g(B)$. The operator A is unique up to a.e. bijections, i.e., for every operator A' satisfying items* (i) *and* (ii), *there exists an a.e. one-to-one function v such that $A' = v(A)$. Moreover, A is a generator of the von Neumann algebra generated by $\{F(\Delta),\ \Delta \in \mathcal{B}(\mathbb{R})\}$.*

An alternative proof of item (i) is given in [16, 17].

Definition 5 The operator A defined by Theorem 1 (and the corresponding PV measure E) is called the sharp reconstruction of F.

Theorem 1 suggests to interpret [2, 3, 10] the outcomes of the measurement of the unsharp observable F as deriving from a randomization of the outcomes of the measurement of its sharp reconstruction E. Indeed, for every $\Delta \in \mathcal{B}(\mathbb{R})$ and $\lambda \in \sigma(A)$, $\mu^A_{(\Delta)}(\lambda)$ can be interpreted as the probability that the outcome of a measurement of F is in Δ when the outcome of the measurement of E is λ.

An important characterization of POV measures, not necessarily commutative, has been given by Neumark.

Theorem 2 (Neumark [21–23]) *Let F be a POV measure of the Hilbert space $\mathcal{H}$. There exist a Hilbert space $\mathcal{H}^+ \supset \mathcal{H}$ and a PV measure E^+ of the space $\mathcal{H}^+$ such that, $F(\Delta) = P^+E^+(\Delta)_{|\mathcal{H}}$, where P^+ is the operator of projection onto $\mathcal{H}$.*

Definition 6 Each operator $\int f(\lambda) dE^+_\lambda$, where f is a real, one-to-one, measurable function, is said to be a Neumark operator corresponding to F. The Neumark operator $\int \lambda dE^+_\lambda$ is denoted by A^+.

Proposition 1 *Let us consider the extension E^+ of a commutative POV measure F and the Neumark operator $A^+ = \int \lambda d E^+_\lambda$ corresponding to E^+. Let f be a real measurable function which is bounded with respect to E^+. Then, $P^+ f(A^+)_{|\mathcal{H}} = \int f(\lambda)\, dF_\lambda$, and $P^+ f(A^+)_{|\mathcal{H}}$ is a bounded self-adjoint operator.*

In case f is unbounded, the domain of definitions of the operators must be taken into account [18].

Definition 7 Whenever there exists a real bounded one-to-one measurable function $f : \sigma(A^+) \to \mathbb{R}$ such that the sharp reconstruction A of a commutative POV measure F is equivalent to $P^+ f(A^+)_{|\mathcal{H}}$ we write $A \leftrightarrow \Pr A^+$ and say that the sharp reconstruction A is equivalent to the projection of a Neumark operator corresponding to F.

Theorem 3 ([4–6]) *Let F be a commutative POV measure with discrete spectrum $K \subset \mathbb{R}$. Suppose F_{k_i}, $k_i \in K$, discrete. Let A be the sharp reconstruction of F, and E^+ a Neumark extension of F. Then $A \leftrightarrow \Pr A^+$.*

In the language of von Neumann algebras, Theorem 3 can be restated as follows.

Corollary 1 *Let $\mathcal{A}(E^+)$ be the von Neumann-algebra of bounded self-adjoint operators generated by E^+ and $\mathcal{A}(A)$ the von Neumann algebra generated by A. Then, there exist a generator B^+ of $\mathcal{A}(E^+)$ and a generator B of $\mathcal{A}(A)$ such that $B = P^+ B^+_{|\mathcal{H}}$.*

Example 1 ([6]) The position operator Q is the sharp reconstruction of the approximate position observable $Q_f(\Delta) := \int_{-\infty}^{\infty} (\mathbf{1}_\Delta * |f|^2)(x)\, dQ_x$, $\Delta \in \mathcal{B}(\mathbb{R})$, where, f is a probability density. Moreover, if Q^+ is the Neumark extension of Q_f then, $Q = P^+ Q^+_{|\mathcal{H}}$.

3 On the Informational Content of a Quantum Observable

In this section we analyze the concept of informational content of an observable. The starting point is the fact that, starting from a commutative POV measure F, it is possible to recover the corresponding sharp reconstruction E. This was proved in [10]. We state this result in the form of a theorem.

Theorem 4 ([10]) *There exists an algorithmic procedure which allows the reconstruction of E starting from F.*

As we said above, Theorem 1 suggests to interpret the unsharp observable F as the randomization of the sharp observable E. Therefore, we can interpret Theorem 4 as suggesting that there should be a kind of information contained in E which is not lost during the randomization process. This means that, in some sense, F and E should have the same informational content. Therefore, we look for an equivalence relation between observables which capture this kind of informational content. The necessary condition which such a relation must satisfy is that E and F must be equivalent.

In the present section, we recall some well known partial order relations on the set of observables and show that no one of them satisfies this condition. Therefore, we propose a partial ordering for which E and F are always equivalent and establish some relationships between this one and the others well known partial order relations.

Definition 8 (Smearing) Let F_1 and F_2 be two observables. If there exists a Markov kernel $\mu_{(\cdot)}(\lambda)$ such that, $F_1(\Delta) = \int \mu_\Delta(\lambda) F_2(d\lambda)$, we say that F_1 is a smearing of F_2 and write $F_1 \preceq_f F_2$. If $F_1 \preceq_f F_2 \preceq_f F_1$, we say that F_1 and F_2 are $\sim_f$ equivalent and write $F_1 \sim_f F_2$.

Theorem 5 ([16]) *A PV measure E is a smearing of a POV measure F if and only if the range of E is contained in the range of F,* $\mathcal{R}(E) \subset \mathcal{R}(F)$.

Theorem 6 ([12]) *Let E and F be respectively a PV measure and a POV measure. Then,* $\mathcal{R}(E) \subset \mathcal{R}(F)$ *if and only if there exists a measurable function f such that* $E(\Delta) = F(f^{-1}(\Delta))$.

Notice that, if E_1 and E_2 are two PV measures, $E_1 \preceq_f E_2$ if and only if there exists a measurable function f such that $E_1(\Delta) = E_2(f^{-1}(\Delta))$ (see Theorems 5 and 6). This is equivalent to the fact that $A_1 = f(A_2)$ where, A_1 and A_2 are the self-adjoint operators corresponding to E_1 and E_2 respectively.

In the following, we denote by $\mathcal{T}_1^+(\mathcal{H})$ the space of trace class operators with trace one on the Hilbert space $\mathcal{H}$. The states of a system are represented by operators in $\mathcal{T}_1^+(\mathcal{H})$.

Definition 9 Let ρ_1 and ρ_2 be two states. Let F be an observable. If there exists a set $\Delta \in \mathcal{B}(\mathbb{R})$ such that $\mathrm{Tr}[F(\Delta)\rho_1] \neq \mathrm{Tr}[F(\Delta)\rho_2]$ we say that F can distinguish between the states ρ_1 and ρ_2.

Definition 10 (State distinction) If for all ρ_1, ρ_2

$$\mathrm{Tr}[F_2(\Delta)\rho_1] = \mathrm{Tr}[F_2(\Delta)\rho_2], \quad \forall \Delta \in \mathcal{B}(\mathbb{R})$$

$$\Downarrow$$

$$\mathrm{Tr}[F_1(\Delta)\rho_1] = \mathrm{Tr}[F_1(\Delta)\rho_2], \quad \forall \Delta \in \mathcal{B}(\mathbb{R})$$

we say that the state distinction power of F_2 is greater than or equal to F_1 and write $F_1 \preceq_i F_2$. If $F_1 \preceq_i F_2 \preceq_i F_1$, we say that F_1 and F_2 are $\sim_i$ equivalent or that they have the same informational content and write $F_1 \sim_i F_2$.

Definition 11 The set of the states determined by the observable F is $\mathcal{O}_F := \{\rho \mid \forall \rho' \neq \rho,\ \exists \Delta,\ \mathrm{Tr}[F(\Delta)(\rho - \rho')] \neq 0\}$.

Definition 12 (State determination) Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be the sets of states determined by F_1 and F_2 respectively. If $\mathcal{O}_1 \subset \mathcal{O}_2$ we say that F_2 has a state determination power greater or equal than F_1 and write $F_1 \preceq_d F_2$. If $F_1 \preceq_d F_2 \preceq_d F_1$, we say that F_1 and F_2 are $\sim_d$ equivalent and write $F_1 \sim_d F_2$.

Theorem 7 ([13]) $F_1 \preceq_f F_2 \Rightarrow F_1 \preceq_i F_2 \Rightarrow F_1 \preceq_d F_2$.

Now we prove that all the equivalence relations between observables introduced above do not ensure that a commutative POV measure is equivalent to its sharp reconstruction. We begin with the concept of smearing. The following example shows that the sharp reconstruction E of a commutative POV measure F need not be a smearing of F, while F is always a smearing of E (see Theorem 1).

Example 2 Let us consider a physical system with spin $J = 1$. The corresponding Hilbert space is $\mathcal{H} = \mathbb{C}^3$. Let E_{-1}, E_0, E_1 be the projection operators corresponding to the eigenvectors of the spin observable $J_3 = \sum_{m=-1}^{1} m E_m$. Let us consider the POV measure $F : \{1, 2, 3\} \to \{F_1, F_2, F_3\}$ where,

$$F_1 = 1/2E_{-1} + 1/2E_0 + 1/4E_1, \qquad F_2 = 1/5E_{-1} + 1/5E_0 + 1/4E_1,$$
$$F_3 = 3/10E_{-1} + 3/10E_0 + 1/2E_1.$$

The sharp reconstruction of F is the PV measure $E : \{1, 2\} = [(E_{-1} + E_0), E_1]$. Since E_1, $(E_{-1} + E_0) \notin \mathcal{R}(F)$, the range of E is not contained in the range of F. Therefore, by Theorem 5, $E \neq_f F$.

Now, we proceed analogously for the concepts of state distinction and state determination.

Example 3 Let $\{|\psi_i\rangle\}_{i\in\mathbb{N}}$ be a basis of $\mathcal{H}$. Let $E_i = |\psi_i\rangle\langle\psi_i|$ be the projector corresponding to $|\psi_i\rangle$. Let us consider the POV measure

$$F(\Delta) = \begin{cases} \Phi & \text{if } 1 \in \Delta \text{ and } 0 \notin \Delta, \\ C = I - \Phi & \text{if } 1 \notin \Delta \text{ and } 0 \in \Delta, \\ \mathbf{I} & \text{if } 1, 0 \in \Delta, \\ \mathbf{0} & \text{if } 1 \notin \Delta \text{ and } 0 \notin \Delta, \end{cases}$$

where,

$$\Phi = \sum_{i=1}^{\infty} \lambda_i E_i = 1/2E_1 + 1/3E_2 + 2/3E_3 + \sum_{i=4}^{\infty} 1/i\, E_i.$$

The sharp reconstruction of F is (see [4]) $E(\Delta) = \sum_{\lambda_i \in \Delta} E_i$. Let us consider the states $\rho_1 = E_1$ and $\rho_2 = 1/2(E_2 + E_3)$. We have,

$$\mathrm{Tr}[\rho_1 F(\Delta)] = \mathrm{Tr}[\rho_2 F(\Delta)], \quad \forall \Delta \subset \mathbb{R}$$

while,

$$\mathrm{Tr}[\rho_1 E((1/3, 1/2])] = 1, \qquad \mathrm{Tr}[\rho_2 E((1/3, 1/2]) = 0$$

so that, $E \preceq_i F$ *is false*.

Notice that, $\rho \in \mathcal{O}_E$ if and only if ρ is a one dimensional spectral projection of E (see [9]) so that $\rho_1 \in \mathcal{O}_E$, while $\rho_1 \notin \mathcal{O}_F$. Therefore, $E \preceq_d F$ *is false*.

Now, we introduce an equivalence relation between observables which should capture the meaning of equivalence between a commutative POV measure and its sharp reconstruction outlined above. Moreover, the relation we are going to introduce is not restricted to commutative POV measures. Some suggestions for its physical interpretation are given at the end of the present section.

Definition 13 Let F_1 and F_2 be two POV measures. We say that $F_1 \preceq_a F_2$ if, for each real, bounded, measurable function f there exists a real, bounded, measurable function g_f such that

$$B_1 := P^+ f(A_1^+)_{|\mathcal{H}} = \int f\, dF_1 \preceq_f \int g_f\, dF_2 = P^+ g_f(A_2^+)_{|\mathcal{H}} =: B_2.$$

If $F_1 \preceq_a F_2 \preceq_a F_1$ we say that F_1 and F_2 are $\sim_a$ equivalent and write $F_1 \sim_a F_2$.

Notice that, B_1 and B_2 are self-adjoint operators, so that $B_1 \preceq_f B_2$ means that there exists a measurable function h such that $B_1 = h(B_2)$ (see the comment to Theorems 5 and 6). The following theorem shows that, at least in the hypothesis of Theorem 3, a commutative POV measure is always equivalent to its sharp reconstruction in the sense of Definition 13.

Theorem 8 *Let F be a commutative POV measure with discrete spectrum such that the operators in the range of F are discrete. Let E be its sharp reconstruction. Then, $E \sim_a F$.*

Proof Let $A = \int \lambda\, dE_\lambda$ be the self-adjoint operator corresponding to E. By Theorem 1, $\mathcal{A}(E) = \mathcal{A}(A) = \mathcal{A}(F)$, where $\mathcal{A}(A)$ and $\mathcal{A}(F)$ are the von Neumann algebras generated by A and F respectively. Notice that, for each real bounded (with respect to F) measurable function f, the self-adjoint operator $\int h(t)F(dt)$ is contained in $\mathcal{A}(F)$. Therefore, for each real, bounded and measurable function h, there exists a function G_h such that $\int h(t)F(dt) = G_h(A)$. This means that $F \preceq_a E$. Moreover, by Theorem 3, there exist two one-to-one functions f and G_f such that $G_f(A) = \int f\, F(dt)$. Since $G_f(A)$ is a generator of $\mathcal{A}(A)$, one has, $\int h dE \preceq_f \int f(t)F(dt)$, for each real bounded, measurable function h. This means that $E \preceq_a F$. □

Now, we point out some relationships between Definition 13 and the other partial order relations on the space of observables introduced above.

Theorem 9 *Let F_1 and F_2 be two POV measures. If $F_1 \preceq_f F_2$, then $F_1 \preceq_a F_2$.*

Proof Since $F_1 \preceq_f F_2$, there exists a Markov kernel $\mu_{(\cdot)}(\lambda)$ such that

$$F_1([t, t+dt)) = \int_{-\infty}^{+\infty} \mu_{[t,t+dt)}(\lambda) F_2(d\lambda).$$

Then, for each $x \in \mathcal{H}$,

$$\begin{aligned}
&\int_{-\infty}^{+\infty} f(t)\langle F_1([t, t+dt))x, x\rangle \\
&= \int_{-\infty}^{+\infty} f(t) \int_{-\infty}^{+\infty} \mu_{[t,t+dt)}(\lambda)\langle F_2(d\lambda)x, x\rangle \\
&= \int_{-\infty}^{+\infty} \langle F_2(d\lambda)x, x\rangle \int_{-\infty}^{+\infty} f(t)\, \mu_{[t,t+dt)}(\lambda) = \int_{-\infty}^{+\infty} g_f(\lambda)\langle F_2(d\lambda)x, x\rangle \qquad (3)
\end{aligned}$$

where, f is a real bounded and measurable function whose infimum and supremum are denoted by m and M respectively and $g_f(\lambda) := \int f(t)\, \mu_{[t,t+dt)}(\lambda) \le M$. Therefore, by the polarization identity, $\int f\, dF_1 = \int g_f\, dF_2$ so that, $F_1 \preceq_a F_2$.

In order to justify the change in the order of integration in equation (3) we proceed as follows. First, we notice that

$$\omega(\cdot)=\int_{-\infty}^{+\infty}\mu_{(\cdot)}(\lambda)\langle F_2(d\lambda)x,x\rangle=\langle F_1(\cdot)x,x\rangle$$

is, for every $x\in\mathcal{H}$, a Lebesgue-Stieltjes measure. Therefore, by the definition of Lebesgue-Stieltjes integral [14],

$$\begin{aligned}\int_{-\infty}^{+\infty}f(t)\omega(dt)&=\lim_{\substack{n\to\infty\\|\delta_n|\to 0}}\sum_{k=1}^{n}f_{k-1}^{(n)}\omega\Big\{t\in\mathbb{R}:f(t)\in(f_{k-1}^{(n)},f_k^{(n)}]\Big\}\\&=\lim_{\substack{n\to\infty\\|\delta_n|\to 0}}\sum_{k=1}^{n}f_{k-1}^{(n)}\int_{-\infty}^{+\infty}\mu_{(E_{k-1}^{(n)})}(\lambda)\langle F_2(d\lambda)x,x\rangle\\&=\lim_{\substack{n\to\infty\\|\delta_n|\to 0}}\int_{-\infty}^{+\infty}\sum_{k=1}^{n}f_{k-1}^{(n)}\mu_{(E_{k-1}^{(n)})}(\lambda)\langle F_2(d\lambda)x,x\rangle\end{aligned}$$

where it was introduced a sequence of subdivisions $\delta_n=\{[f_0,f_1^{(n)}],(f_1^{(n)},f_2^{(n)}],\dots,(f_{n-1}^{(n)},f_n]\}$, $m=f_0<f_1<\cdots<f_n=M$, of the interval $[m,M]$, such that $|\delta_n|=\max_{1\le k\le n}\{(f_k^{(n)}-f_{k-1}^{(n)})\}\to 0$, when $n\to\infty$, and it was set $E_{k-1}^{(n)}=\{t\in\mathbb{R}:f(t)\in(f_{k-1}^{(n)},f_k^{(n)}]\}$.

Now let us consider the sequence of functions $H_n(\lambda)=\sum_{k=1}^{n}f_{k-1}^{(n)}\mu_{(E_{k-1}^{(n)})}(\lambda)$. One has $H_n(\lambda)\le\sup\{|f|\}\mu_{(\mathbb{R})}(\lambda)=M<\infty$, for each $\lambda\in\mathbb{R}$ and $n\in\mathbb{N}$.

Moreover, by the integrability of f with respect to the measure $\mu_{(\cdot)}(\lambda)$,

$$\lim_{n\to\infty}H_n(\lambda)=\int_{-\infty}^{+\infty}f(t)\,\mu_{[t,t+dt)}(\lambda)=g_f(\lambda).$$

By Theorem 11 in [7],

$$\begin{aligned}&\lim_{n\to\infty}\int_{-\infty}^{+\infty}\sum_{k=1}^{n}f_{k-1}^{(n)}\mu_{(E_{k-1}^{(n)})}(\lambda)\langle F_2(d\lambda)x,x\rangle\\&=\int_{-\infty}^{+\infty}\lim_{n\to\infty}H_n(\lambda)\langle F_2(d\lambda)x,x\rangle\\&=\int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty}f(t)d_t\mu_t(\lambda)\right]\langle F_2(d\lambda)x,x\rangle=\int_{-\infty}^{+\infty}g_f(\lambda)\langle F_2(d\lambda)x,x\rangle.\end{aligned}$$ □

Up to now, we have proved that $F_1\preceq_f F_2$ implies $F_1\preceq_a F_2$ so that, smearing is stronger than Definition 13. Moreover, it is worth remarking that neither state distinction nor state determination are weaker than Definition 13. This can be seen by resorting to Example 3 where, $E\sim_a F$ while, $E\preceq_i F$ and $E\preceq_d F$ were proved to be false.

We end this section with some observations on the physical meaning of Definition 13. Let us consider the sets $A_i=\{B_f:=\int f\,dF_i,\ f\text{ bounded and measurable}\}$, $i=1,2$. The operators in A_i, represent a set of compatible sharp observables. They are named compatible since the expectation values of each operator in A_i are determined by the same unsharp observable F_i by means of the relation

$$\mathrm{Tr}[\rho B_f] = \mathrm{Tr}\left[\rho \int f dF_i\right] = \int f(t)\, \mathrm{Tr}[\rho F_i(dt)].$$

We remark that the knowledge of the expectation values $\mathrm{Tr}[\rho B]$ of a self-adjoint operator B, for each state ρ, determines the spectral measure E^B corresponding to B. Moreover, if g is a real bounded and measurable function, then $E^{g(B)}(\Delta) = E^B(g^{-1}(\Delta))$, for any Borel set Δ. Therefore, the knowledge of the spectral measure E^B of B determines the spectral measure $E^{g(B)}$ of $g(B)$. All that suggests (see also [5]) to generalize the definition of a compatible set of observables corresponding to an unsharp observable F by including also the observables of the kind $g(B_f)$, with $B_f = \int f dF$ and g bounded and measurable. In other words, we propose to define the set of compatible sharp observables corresponding to the unsharp observable F by the set $\{B_f := \int f dF,\ f$ bounded and measurable$\} \cup \{g(B_f),\ g$ bounded and measurable$\}$. By adopting this slightly different definition of a compatible set, we can say that the observable F_2 in Definition 13 is such that its set of compatible observables is bigger than the set of compatible observables corresponding to F_1.

This can be a good starting point toward a physical interpretation of Definition 13 but this problem deserves a future work for a more deep analysis.

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